

## KOLLÁR–NADEL TYPE VANISHING THEOREM

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ABSTRACT. We prove an analytic generalization of Kollár’s vanishing theorem, which contains the Nadel vanishing theorem as a special case.

## 1. INTRODUCTION

In the conference, I talked about the Hodge theoretic aspect of injectivity and vanishing theorems (see [2], [3], and [4]). Here, I will explain some analytic generalizations. In [5], Shin-ichi Matsumura and I established the following theorems.

**Theorem 1.1** ([5, Theorem A]). *Let  $F$  be a holomorphic line bundle on a compact Kähler manifold  $X$  and let  $h$  be a singular hermitian metric on  $F$ . Let  $M$  be a holomorphic line bundle on  $X$  equipped with a smooth hermitian metric  $h_M$ . We assume that*

$$\sqrt{-1}\Theta_{h_M}(M) \geq 0 \quad \text{and} \quad \sqrt{-1}\Theta_h(F) - a\sqrt{-1}\Theta_{h_M}(M) \geq 0$$

*for some  $a > 0$ . Let  $s$  be a nonzero global section of  $M$ . Then the map*

$$\times s : H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h)) \rightarrow H^i(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes M)$$

*induced by  $\otimes s$  is injective for every  $i$ , where  $\omega_X$  is the canonical bundle of  $X$  and  $\mathcal{J}(h)$  is the multiplier ideal sheaf of  $h$ .*

Theorem 1.1 is a generalization of Enoki’s injectivity theorem (see [1, Theorem 0.2]). Although the formulation of Theorem 1.1 may look artificial, it has many interesting applications (see [5]). Theorem 1.2 below is a Bertini-type theorem for multiplier ideal sheaves.

**Theorem 1.2** ([5, Theorem 1.10]). *Let  $X$  be a compact complex manifold, let  $\Lambda$  be a free linear system on  $X$  with  $\dim \Lambda \geq 1$ , and let  $\varphi$  be a quasi-plurisubharmonic function on  $X$ . We put*

$$\mathcal{G} = \{H \in \Lambda \mid H \text{ is smooth and } \mathcal{J}(\varphi|_H) = \mathcal{J}(\varphi)|_H\}.$$

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Then  $\mathcal{G}$  is dense in  $\Lambda$  in the classical topology. Note that  $\mathcal{J}(\varphi)$  is the multiplier ideal sheaf of  $\varphi$ .

The main purpose of this paper is to prove the following theorem, which is a slight generalization of [5, Theorem D], as an application of Theorem 1.1 and Theorem 1.2.

**Theorem 1.3** (Vanishing theorem of Kollár–Nadel type). *Let  $f : X \rightarrow Y$  be a holomorphic map from a compact Kähler manifold  $X$  to a projective variety  $Y$ . Let  $F$  be a holomorphic line bundle on  $X$  equipped with a singular hermitian metric  $h$ . Let  $H$  be an ample line bundle on  $Y$ . Assume that there exists a smooth hermitian metric  $g$  on  $f^*H$  such that*

$$\sqrt{-1}\Theta_g(f^*H) \geq 0 \quad \text{and} \quad \sqrt{-1}\Theta_h(F) - \varepsilon\sqrt{-1}\Theta_g(f^*H) \geq 0$$

for some  $\varepsilon > 0$ . Then we have

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = 0$$

for every  $i > 0$  and  $j$ , where  $\omega_X$  is the canonical bundle of  $X$  and  $\mathcal{J}(h)$  is the multiplier ideal sheaf associated to the singular hermitian metric  $h$ .

We can easily see that Theorem 1.3 contains Demailly's original formulation of the Nadel vanishing theorem (see [5, Theorem 1.4]) and Kollár's vanishing theorem (see [6, Theorem 2.1 (iii)]) as special cases. Therefore, we call Theorem 1.3 the vanishing theorem of Kollár–Nadel type. For a related vanishing theorem, see [7, Theorem 1.3].

In this paper, we will freely use the same notation as in [5].

## 2. PROOF OF THEOREM 1.3

In this section, we prove Theorem 1.3 as an application of Theorem 1.1 and Theorem 1.2. I hope that the following proof will show the reader how to use Theorem 1.1 and Theorem 1.2.

*Proof of Theorem 1.3.* We use the induction on  $\dim Y$ . If  $\dim Y = 0$ , then the statement is obvious. We take a sufficiently large positive integer  $m$  and a general member  $B \in |H^{\otimes m}|$  such that  $D = f^{-1}(B)$  is smooth, contains no associated primes of  $\mathcal{O}_X/\mathcal{J}(h)$ , and satisfies  $\mathcal{J}(h|_D) = \mathcal{J}(h)|_D$  by Theorem 1.2. By the Serre vanishing theorem, we may further assume that

$$(2.1) \quad H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) = 0$$

for every  $i > 0$  and  $j$ . We have the following big commutative diagram.

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \mathcal{J}(h) \otimes \mathcal{O}_X(-D) & \xrightarrow{\alpha} & \mathcal{J}(h) & \longrightarrow & \text{Coker } \alpha \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \beta \\
 0 & \longrightarrow & \mathcal{O}_X(-D) & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_D \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \\
 & & (\mathcal{O}_X/\mathcal{J}(h)) \otimes \mathcal{O}_X(-D) & \xrightarrow{\gamma} & \mathcal{O}_X/\mathcal{J}(h) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 
 \end{array}$$

Since  $D$  contains no associated primes of  $\mathcal{O}_X/\mathcal{J}(h)$ ,  $\gamma$  is injective. This implies that  $\beta$  is injective by the snake lemma and that  $\text{Coker } \alpha = \mathcal{J}(h)|_D = \mathcal{J}(h|_D)$ . Thus we obtain the following short exact sequence:

$$0 \rightarrow \mathcal{J}(h) \otimes \mathcal{O}_X(-D) \rightarrow \mathcal{J}(h) \rightarrow \mathcal{J}(h|_D) \rightarrow 0.$$

By taking  $\otimes \omega_X \otimes F \otimes \mathcal{O}_X(D)$  and using adjunction, we obtain the short exact sequence:

$$0 \rightarrow \omega_X \otimes F \otimes \mathcal{J}(h) \rightarrow \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^* H^{\otimes m} \rightarrow \omega_D \otimes F|_D \otimes \mathcal{J}(h|_D) \rightarrow 0.$$

Therefore, we see that

$$\begin{aligned}
 (2.2) \quad 0 &\rightarrow R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \rightarrow R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m} \\
 &\rightarrow R^j f_*(\omega_D \otimes F|_D \otimes \mathcal{J}(h|_D)) \rightarrow 0
 \end{aligned}$$

is exact for every  $j$  since  $B$  is a general member of  $|H^{\otimes m}|$ . By induction on  $\dim Y$ , we have

$$(2.3) \quad H^i(B, R^j f_*(\omega_D \otimes F|_D \otimes \mathcal{J}(h|_D))) = 0$$

for every  $i > 0$  and  $j$ . By taking the long exact sequence associated to (2.2), we obtain

$$H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m})$$

for every  $i \geq 2$  and  $j$  by (2.3). Thus we have

$$(2.4) \quad H^i(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = 0$$

for every  $i \geq 2$  and  $j$  by (2.1). By Leray's spectral sequence and (2.1) and (2.4), we have the following commutative diagram:

$$\begin{array}{ccc} H^1(Y, \mathcal{F}^j) & \hookrightarrow & H^{j+1}(X, \omega_X \otimes F \otimes \mathcal{J}(h)) \\ \alpha \downarrow & & \downarrow \beta \\ H^1(Y, \mathcal{F}^j \otimes H^{\otimes m}) & \hookrightarrow & H^{j+1}(X, \omega_X \otimes F \otimes \mathcal{J}(h) \otimes f^* H^{\otimes m}) \end{array}$$

for every  $j$ , where  $\mathcal{F}^j = R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))$ . Note that the horizontal arrows are injective. Since  $\beta$  is injective by Theorem 1.1, we obtain that  $\alpha$  is also injective. By (2.1), we have

$$H^1(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h)) \otimes H^{\otimes m}) = 0$$

for every  $j$ . Therefore, we see that  $H^1(Y, R^j f_*(\omega_X \otimes F \otimes \mathcal{J}(h))) = 0$  for every  $j$ . Thus we obtain the desired vanishing theorem: Theorem 1.3.  $\square$

We close this section with a remark on Nakano semipositive vector bundles.

*Remark.* Let  $E$  be a Nakano semipositive vector bundle on  $X$ . We can easily see that Theorem 1.3 holds even when  $\omega_X$  is replaced by  $\omega_X \otimes E$ . We leave the details as an exercise for the reader (see [5, Section 6]).

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